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SUFFICIENT CONDITIONS FOR STABILITY "IN THE LARGE" OF A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

by Jerrold H. Suddath

Langley Research Center
Langley Station, Hampton, Va.

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SUFFICIENT CONDITIONS FOR STABILITY "IN THE LARGE" OF

A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

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SUMMARY

An analytical study has been made of a problem proposed by the Russian mathematician, M. A. Aizerman. The problem, which has significance in the theory of automatic control, concerns the stability "in the large" (i.e., every motion converges to a stable-equilibrium state as time approaches infinity) of dynamic systems described by nonlinear differential equations of a particular form. In proposing the problem, Aizerman made an assertion based on a linear analogy to a nonlinear system and stated the desirability of obtaining either proof or refutation of the assertion. The purpose of this paper is to show that Aizerman's assertion is true for cases in which the region of stable gains (for the analogous linear system) has a certain property.

INTRODUCTION

In the study of automatic control systems, it is often desirable to ascertain the conditions under which every motion converges to a stable-equilibrium state as time approaches infinity. The latter situation has been referred to as stability "in the large" by Aizerman (ref. 1), and his terminology will be used hereinafter in the present report. In reference 1 a particular problem of this sort was proposed; an assertion was made based on a linear analogy of the nonlinear system, and it was deemed desirable to obtain either proof or refutation of the assertion. Special cases and examples of this problem in the theory of automatic control systems were treated in reference 2.

The purpose of this paper is to show that the assertion is true for cases in which the region of stable gains (for the analogous linear system) has a certain property. In reference 1 it was pointed out that Aizerman was unable to prove his assertion by the Second Method of Lyapunov. This negative result suggested that a different approach should be taken in the analysis, and the proof presented herein is based on an application of Gronwall's lemma (see ref. 3), rather than the Second Method of Lyapunov.

SYMBOLS

$[A]$ n by n matrix of constant coefficients

$[\hat{A}] \triangleq [A] + \frac{1}{2}(\alpha + \beta)\vec{e}_1\vec{e}_k{}^*$

a	parametric control sensitivity or gain
a_{ij}	element in ith row and jth column of $[A]$
c	constant associated with norm of $e^{[\hat{A}]t}$
$\vec{e}_i$	ith column of identity matrix
$e^{[\hat{A}]t}$	n by n transition matrix
$f(x_k), g(x_k)$	scalar functions of x_k
$\vec{x}$	n-dimensional state vector
$\|\vec{x}\|$	norm of vector $\vec{x}$, $\sum_{i=1}^{n} \lvert x_i \rvert$
$\left\| e^{[\hat{A}]t} \right\|$	norm of matrix $e^{[\hat{A}]t}$, equal to sum of absolute values of matrix elements
α	lower stability limit on a
$\alpha' = \alpha + \delta$	
β	upper stability limit on a
$\beta' = \beta - \delta$	
δ	defined by $0 < \delta < \frac{1}{2}(\beta - \alpha)$
λ	minimum of absolute values of real parts of eigenvalues of $[\hat{A}]$

Subscript:

o	initial condition

A dot over a symbol denotes differentiation with respect to time. An arrow above a symbol denotes a vector.

ANALYSIS

Statement of Problem

Consider the system of linear differential equations,

$$\left.\begin{aligned} \dot{x}_1 &= \sum_{j=1}^{n} a_{1j}x_j + ax_k \\ \dot{x}_i &= \sum_{j=1}^{n} a_{ij}x_j \qquad (i = 2, 3, \ldots n) \end{aligned}\right\} \tag{1}$$

where k is some integer between 1 and n.

For the given constants a_{ij}, i, $j = 1, 2, \ldots n$ and for any value of a in the interval $\alpha < a < \beta$, let all the roots of the characteristic equation of system (1) have negative real parts.

Assertion.- The nonlinear system (analogous to system (1)) given by

$$\left.\begin{aligned} \dot{x}_1 &= \sum_{j=1}^{n} a_{1j}x_j + f(x_k) \\ \dot{x}_i &= \sum_{j=1}^{n} a_{ij}x_j \qquad (i = 2, 3, \ldots n) \end{aligned}\right\} \tag{2}$$

is asymptotically stable in the large if the origin is the only equilibrium point and if $f(x_k)$ is any single-valued continuous function that satisfies the following conditions:

$$\alpha x_k^2 < x_k f(x_k) < \beta x_k^2$$

for all $x_k \neq 0$ and

$$f(0) = 0$$

The problem here is that of finding conditions under which the assertion is true.

Reformulation of problem.- In attacking this problem, it was found convenient to make the following definitions and changes in notation:

$$g(x_k) = f(x_k) - \frac{1}{2}(\alpha + \beta)x_k$$

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix}$$

$$[A] = [a_{ij}]$$

With these definitions, system (2) can be written as

$$\dot{\vec{x}} = [A]\vec{x} + \vec{e}_1\left[\frac{1}{2}(\alpha + \beta)x_k + g(x_k)\right] \tag{3}$$

Now, define $[\hat{A}]$ by

$$[\hat{A}] \triangleq [A] + \frac{1}{2}(\alpha + \beta)\vec{e}_1\vec{e}_k{}^*$$

where $\vec{e}_k{}^*$ is the kth row of the identity matrix.

Then equation (3) can be written as

$$\dot{\vec{x}} = [\hat{A}]\vec{x} + \vec{e}_1 g(x_k) \tag{4}$$

where, by definition, all the eigenvalues of $[\hat{A}]$ have negative real parts and $g(x_k)$ satisfies $g(0) = 0$, and where

$$\frac{1}{2}(\alpha - \beta)x_k{}^2 < x_k\, g(x_k) < \frac{1}{2}(\beta - \alpha)x_k{}^2$$

For brevity, let

$$K = \frac{1}{2}(\beta - \alpha) > 0$$

and note that

$$\left|g(x_k)\right| \leqq K\left|x_k\right| \leqq K\left\|\vec{x}\right\|$$

where

$$\left\|\vec{x}\right\| \equiv \sum_{i=1}^{n} \left|x_i\right|$$

Furthermore, note that there exist positive constants c and λ, such that

$$\left\|e^{\left[\hat{A}\right]t}\right\| \leqq ce^{-\lambda t}$$

Derivation of Sufficient Conditions

It is well known that the solution of equation (4) can be written as follows (see ref. 3, for example):

$$\vec{x}(t) = e^{\left[\hat{A}\right]t}\vec{x}_0 + \int_0^t e^{\left[\hat{A}\right](t-s)}\vec{e}_1\, g\left(x_k(s)\right)ds$$

which, from elementary inequalities of norms, leads to

$$\left\|\vec{x}(t)\right\| \leqq c^* e^{-\lambda t} + Kce^{-\lambda t}\int_0^t e^{\lambda s}\left\|\vec{x}(s)\right\| ds \tag{5}$$

where $c^* \equiv c\left\|\vec{x}_0\right\|$. From equation (5) it follows that

$$e^{\lambda t}\left\|\vec{x}(t)\right\| \leqq c^* + Kc\int_0^t e^{\lambda s}\left\|\vec{x}(s)\right\| ds \tag{6}$$

A version of Gronwall's lemma, which is given in the appendix, can be applied to equation (6) to obtain

$$e^{\lambda t}\|\vec{x}(t)\| \leqq c^* e^{Kct}$$

or

$$\|\vec{x}(t)\| \leqq c^* e^{-(\lambda - Kc)t}$$

which shows that Aizerman's assertion is certainly true if $\lambda - Kc > 0$ or $K < \lambda/c$; or, in terms of α and β,

$$\beta < \alpha + 2\lambda/c \tag{7}$$

It should be recalled that λ/c depends on $\frac{1}{2}(\alpha + \beta)$.

REMARKS

It should be pointed out that inequality (7) shows that it is possible to select intervals (α',β') such that the assertion holds if $f(x_k)$ satisfies the conditions $\alpha' x_k^2 < x_k f(x_k) < \beta' x_k^2$ and $f(0) = 0$. This result follows from the fact that if inequality (7) were not satisfied by α and β, the entire analysis could be repeated with α and β replaced by

$$\alpha' = \alpha + \delta$$

and

$$\beta' = \beta - \delta$$

respectively, where δ satisfies the condition that

$$0 < \delta < \frac{1}{2}(\beta - \alpha)$$

and condition (7) would be replaced by $\beta' < \alpha' + 2\lambda/c$, which could be satisfied by proper choice of δ. (Note that λ/c does not change.) This result is in agreement with the results stated in reference 1, which also points out that $\alpha < \alpha' < \beta' < \beta$ so that the intervals (α',β') form only part of the interval (α,β).

The result of this analysis is the fact that Aizerman's assertion is true for systems with constants α and β satisfying inequality (7).

CONCLUDING REMARKS

An analytical study has been made of a problem proposed by the Russian mathematician, M. A. Aizerman, which concerns the stability "in the large" (i.e., every motion converges to a state of stable equilibrium as time approaches infinity) of dynamic systems described by nonlinear differential equations of a particular form. This problem has significance in the theory of automatic control systems. By the application of Gronwall's lemma, rather than the Second Method of Lyapunov, it was shown that with an additional proviso, an assertion regarding this problem is true.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., March 3, 1964.

APPENDIX

A VERSION OF GRONWALL'S LEMMA

A version of Gronwall's lemma (see ref. 3) is presented here. Suppose the continuous function $\Phi(t)$, for $t \geqq 0$, satisfies

$$0 \leqq \Phi(t) \leqq c + m\int_0^t \Phi(s)ds \tag{A1}$$

where c and m are positive constants; then $\Phi(t) \leqq ce^{mt}$.

Proof: From equation (A1), it follows that

$$\frac{\Phi(t)}{c + m\int_0^t \Phi(s)ds} \leqq 1$$

or

$$\frac{1}{m}\frac{d}{dt}\left\{\log_e\left[c + m\int_0^t \Phi(s)ds\right]\right\} \leqq 1 \tag{A2}$$

Integrating both sides of equation (A2) from 0 to t gives

$$\log_e\left[\frac{c + m\int_0^t \Phi(s)ds}{c}\right] \leqq mt$$

or

$$c + m\int_0^t \Phi(s)ds \leqq ce^{mt} \tag{A3}$$

From equations (A1) and (A3) it follows that

$$\Phi(t) \leqq ce^{mt}$$

as was to be shown.

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